Linear systems, row reduction and echelon forms: Part 2

We now illustrate the row reduction algorithm to find the reduced echelon form of any matrix and how this allows for the solving of linear systems.

Example: Solve the following linear system:

$$\begin{aligned} 3x_2 - 6x_3 + 4x_4 &= -5 \\ 3x_1 - 7x_2 + 8x_3 + 8x_4 &= 9 \\ 3x_1 - 9x_2 + 12x_3 + 6x_4 &= 15 \end{aligned}$$

Step 0 Write the augmented matrix for the system.

$$\begin{pmatrix} 0 & 3 & -6 & 4 & -5 \\ 3 & -7 & 8 & 8 & 9 \\ 3 & -9 & 12 & 6 & 15 \end{pmatrix}$$

Step 1 Begin with the leftmost column (this is a pivot column). Select a nonzero entry in this column as a pivot and, if necessary, interchange rows to move this entry into the pivot position.

pivot

$R1 \leftrightarrow R3$

$$\begin{pmatrix} 3 & -9 & 12 & 6 & 15 \\ 3 & -7 & 8 & 8 & 9 \\ 0 & 3 & -6 & 4 & -5 \end{pmatrix}$$

Step 2 Use elementary row operations to create zeros in all positions below the pivot.

$$\begin{pmatrix} 3 & -9 & 12 & 6 & 15 \\ 3 & -7 & 8 & 8 & 9 \\ 0 & 3 & -6 & 4 & -5 \end{pmatrix}$$

R2-R1

$$\begin{pmatrix} 3 & -9 & 12 & 6 & 15 \\ 0 & 2 & -4 & 2 & -6 \\ 0 & 3 & -6 & 4 & -5 \end{pmatrix}$$

next pivot column

Step 3 Find the leftmost column with a nonzero leading entry in a row not yet containing a pivot. (This is the next pivot column.) Apply steps 1 and 2 to this column.

$R3 - \frac{3}{2}R2$

$$\begin{pmatrix} 3 & -9 & 12 & 6 & 15 \\ 0 & 2 & -4 & 2 & -6 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Step 4 Repeat steps 1 - 3 until there are no more nonzero rows to modify. ✓

$$\begin{pmatrix} 3 & -9 & 12 & 6 & 15 \\ 0 & 2 & -4 & 2 & -6 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

next pivot column has nothing below it, we're done.

Step 5 Beginning with the rightmost pivot and working upward and to the left, use elementary row operations to create zeros above each pivot.

$$\begin{pmatrix} 3 & -9 & 12 & 6 & 15 \\ 0 & 2 & -4 & 2 & -6 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow[R2-2R3]{R1-6R3} \begin{pmatrix} 3 & -9 & 12 & 0 & -9 \\ 0 & 2 & -4 & 0 & -14 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -9 & 12 & 0 & -9 \\ 0 & 2 & -4 & 0 & -14 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{\frac{1}{2}R2} \begin{pmatrix} 3 & -9 & 12 & 0 & -9 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{R1+9R2} \begin{pmatrix} 3 & 0 & -6 & 0 & -72 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 6 & 0 & -72 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Step 6 Use scaling operation to make each pivot 1 and write down the corresponding system of equations.

$$\xrightarrow{\frac{1}{3}R1} \begin{pmatrix} 1 & 0 & 2 & 0 & -24 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{aligned} x_1 + 2x_3 &= -24 \\ x_2 - 2x_3 &= -7 \\ x_4 &= 4 \end{aligned}$$

Step 7 Identify free and basic variables and give a parametric description of the solution set.

Basic

x_1, x_2, x_4

Free

x_4

$$\boxed{\begin{aligned} x_1 &= -2x_3 - 24 \\ x_2 &= 2x_3 - 7 \\ x_4 &= 4 \end{aligned}}$$